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DEFORMATIONS OF BALLS IN SCHIFFER'S CONJECTURE FOR RIEMANNIAN SYMMETRIC SPACES

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ABSTRACT

It is proved that geodesic balls in a Riemannian symmetric space of rank one are *stable* solutions to a free-boundary problem for the Laplace–Beltrami operator with constant Dirichlet–Neumann boundary conditions. This result supports Schiffer's conjecture that balls are the only solutions to the problem. The main ingredient of the proof is a characterization of geodesic balls by the multiplicity of eigenvalues of the Laplace–Beltrami operator.

1. Introduction

The Schiffer conjecture (cf. [Yau, problem 80]) concerns the connection between geometric symmetry of a domain and the solvability of a certain spectral problem for the Laplace operator on this domain.

More precisely, the conjecture is that balls are the only domains with smooth connected boundaries such that there exists an eigenfunction of the Laplace

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operator with Neumann boundary data which is constant on the boundary and non-constant on the domain.

Brown, Schreiber and Taylor [BST] and Berenstein [B] have established a close relation between Schiffer's conjecture and the so-called Pompeiu problem in integral geometry over the motion group (cf. [Z1, Z2, Z3]). L. Zalcman's survey [Z3] contains information on the current state of the problem.

This paper is devoted to the study of deformations of geodesic balls in Riemannian symmetric spaces of rank one, from the point of view of Schiffer's conjecture. We want to know to what extent one can deform a ball, preserving the solvability of a corresponding Dirichlet–Neumann spectral problem for the Laplace–Beltrami operator.

In Theorem 2 we show that, essentially, the only perturbations of a geodesic ball which preserve the solvability of the Dirichlet–Neumann spectral problem are trivial perturbations by balls. Since Schiffer's conjecture is that only balls provide solvability of this spectral problem, Theorem 2 supports this conjecture.

Earlier, a perturbation theorem of this kind was obtained for the Euclidean plane by the first author [A] and, independently, by a different method by T. Kobayashi [Ko] for the Euclidean spaces of any dimensions. Though the proofs in [A] and [Ko] are completely different, both of them strongly use explicit constructions and special formulas from harmonic analysis in Euclidean spaces. The proof in [A] also uses Riemann mappings (and, therefore, works for the case of the plane only) and the proof in [Ko] is based on quite delicate technical estimates for Fourier transforms of characteristic functions of domains.

The main idea in this paper is to exploit the multiplicity of eigenvalues of the Laplace operator. We prove in Theorem 1 that a geodesic ball can be characterized by the multiplicity of the Dirichlet–Neumann spectrum. Then the result concerning deformations in the Dirichlet–Neumann problem follows, due to the standard arguments of perturbation theory that the multiplicity of eigenvalues by small perturbations of operators does not increase. This approach uses, essentially, only the homogeneous structure and enables us, in particular, to extend the results of [A] and [Ko] to Riemannian spaces. Note that the idea of inferring geometric symmetries from the multiplicity of eigenvalues has also been used in works by P. Aviles [Av] and B. Kawohl [Ka], devoted to the Schiffer–Pompeiu problem. Essentially, the paper [Av] contains the proof of Theorem 1 for the Euclidean case.

The sense of Theorem 2, from the point of view of the Pompeiu problem, is that if domains are close enough to a ball, but are not balls, then they have the Pompeiu property, i.e., provide uniqueness in the corresponding integral geometry problem. It is interesting to compare this with Brown and Kahane's result [BK] which says that if a domain is geometrically far from being a ball (a cigar-like domain), then it also possesses the Pompeiu property. We would like to emphasize that our result is valid for spaces of non-positive curvature. In the case of positive curvature there are examples of nonisolated domains that fail the Schiffer conjecture, as was shown by C. Berenstein and P.C. Yang [BY]

The authors thank Carlos Berenstein with whom the idea of deriving geometry from the multiplicity of eigenvalues of the Laplace operator was discussed earlier.

2. Preliminaries and the main results

2.1 Throughout the whole paper, M will denote a noncompact connected globally symmetric Riemannian space of rank 1, $d = \dim M$. We will assume that M is irreducible, so M is of noncompact or Euclidean type. This means that $M = G/K$, where G is a connected real semisimple Lie group with finite center and K is a maximal compact subgroup, and if $M = \mathbb{R}^n$ then $G = M(\mathbb{R}^n)$, the group of all rigid motions, and $K = \mathrm{SO}(n)$.

We denote $o = \pi(e)$, where $\pi: G \to G/K$ is the canonical projection and e is the unit of G. By $B(a, r)$ we denote the geodesic ball $B(a,r) = \{x \in M: \rho(a,x) < r\}$; ρ is the Riemannian metric on M and $B = B(0,1)$.

L will denote the Laplace–Beltrami operator $L = \operatorname{div}\operatorname{grad}$ on M (cf. Helgason [He]).

2.2 For a bounded domain $\Omega \subset M$ with connected smooth boundary $\partial\Omega$ we will consider two kinds of boundary value problems for the operator L, the spectral problem with Dirichlet boundary data

$$(\mathrm{D})_\Omega \qquad \begin{aligned} &L\psi + \mu\psi = 0 \ , \quad \psi \in C^2(\Omega) \cap C^1(\overline{\Omega}), \\ &\psi \,\big|_{\partial\Omega} = 0, \end{aligned}$$

and the spectral problem with constant Dirichlet-Neumann boundary data

$$(\mathrm{DN})_\Omega \qquad \begin{aligned} &Lu + \lambda u = 0 \ , \quad u \in C^2(\Omega) \cap C^1(\overline{\Omega}), \\ &u \,\big|_{\partial\Omega} = c \\ &\operatorname{grad} u \,\big|_{\partial\Omega} = 0, \end{aligned}$$

where c is a constant.

We will call μ a D**-eigenvalue for the domain** Ω if the problem $(\mathrm{D})_\Omega$ has a non-zero solution ψ, and we call λ a DN**-eigenvalue for** Ω if the problem $(\mathrm{DN})_\Omega$ has a *non-constant* eigenfunction u. This is equivalent to $\lambda \neq 0$ and the existence of the non-zero solution u, which in turn is equivalent to $c \neq 0$.

Denote by $m_D(\Omega, \mu)$ and $m_{DN}(\Omega, \lambda)$ the dimensions of the corresponding eigenspaces. According to Holmgren's uniqueness theorem, the multiplicity $m_{DN}(\Omega, \lambda)$ can take only values 0 and 1.

The **Schiffer conjecture** asserts that the existence of at least one DN-eigenvalue λ for Ω ($m_{DN}(\Omega, \lambda) = 1$) implies $\Omega = B(a, r)$ for some $a \in M$, $r > 0$. We prove the following particular result:

THEOREM 1:

(1) *If there exists at least one* DN*-eigenvalue* λ *for* Ω *such that its* D*-multiplicity* $m_D(\Omega, \lambda) \leq \dim M$, *then* Ω *is a geodesic ball,* $\Omega = B(a, r)$.

(2) *Conversely, if* λ *is a* DN*-eigenvalue of* $B(a, r)$ *then* $m_D(B(a, r), \lambda) = \dim M$.

Theorem 1 and arguments of operator perturbation theory enable us to prove the "rigidity" theorem for deformations of geodesic balls by domains with solvable DN-problems.

Before formulating the results we shall give the definition of deformation we wish to consider.

Definition 2.1: We say that a one-parametric family $\{\Omega_t\}_{t\in[0,T)}$ is a DN**-deformation** of the unit ball B if there exists a mapping

$$F\colon [0, T) \times \overline{B} \to M$$

with properties

(1) $F \in C^2([0, T) \times \overline{B})$,

(2) $F(t, \cdot)\colon \overline{B} \to \overline{\Omega}_t$ is a C^2-diffeomorphism,

(3) $F(0, x) = x$ for all $x \in \overline{B}$, that is $\Omega_0 = B$,

(4) for any $t \in [0, T)$ there exists at least one DN-eigenvalue λ_t for Ω_t and $\sup_{t\in[0,T)} \lambda_t < \infty$.

THEOREM 2: *Let* $\{\Omega_t\}_{t\in[0,T)}$ *be a* DN*-deformation of the unit geodesic ball* B. *Then there exists* $\varepsilon > 0$ *such that for each* $t \in [0, \varepsilon)$ *the domain* Ω_t *is a geodesic ball,* $\Omega_t = B(a_t, r_t)$.

Remark: Throughout the paper we call λ an eigenvalue and $u \not\equiv 0$ an eigenfunction for an operator L if $Lu + \lambda u = 0$.

3. Proof of Theorem 1

In this section we characterize the unit geodesic ball B by the multiplicity of DN-eigenvalues.

3.1 The second part of Theorem 1 concerns the D-multiplicity of DN-eigenvalues for geodesic balls and we are going to examine these assertions straightforwardly. It suffices to consider the unit ball B.

Therefore, let λ be a DN-eigenvalue for B and u be the eigenfunction corresponding to λ. Since for any $k \in K$ the composition $u \circ k$ is again a solution to $(\mathrm{DN})_B$, we have, by the uniqueness theorem, $u = u \circ k$ and, therefore, u is a K-invariant eigenfunction of L, i.e., u is a spherical function.

In the case $M = \mathbb{R}^n$, $L = \Delta$ is the ordinary Laplacian in $\mathbb{R}^n$, and we have

$$u(x) = \text{const}\ \frac{J_{(n-2)/2}(\alpha|x|)}{|x|^{(n-2)/2}}$$

where $\alpha^2 = \lambda$ and $J_{n/2}(\alpha) = 0$, according to the Neumann boundary condition.

Due to this condition, any partial derivative of u is a solution to $(\mathrm{D})_B$, so we can produce n D-eigenfunctions

$$u_j(x) = \frac{\partial}{\partial x_j} u(x) = \text{const}\ \frac{J_{n/2}(\alpha|x|)}{|x|^{n/2}}\, x_j, \quad j = 1, \ldots, n.$$

It is well-known that the functions $u_1, \ldots, u_n$, given by this formula, constitute a basis in the space of all solutions to $\Delta\psi + \lambda\psi = 0$ with boundary data $\psi|_{\partial B} = 0$. Thus, $m_D(\lambda, B) = n$.

3.2 In the general case the arguments are similar. Let us consider the Cartan decomposition

$$\mathfrak{G} = \mathfrak{K} \oplus \mathcal{P}, \quad [\mathfrak{K}, \mathcal{P}] \subset \mathcal{P},$$

of the Lie algebra $\mathfrak{G}$ of G as the direct sum of the Lie algebra of K and the complementary linear space $\mathcal{P}$. The linear space $\mathcal{P}$ is isomorphic to the tangent space $(M)_o$ by the exponential mapping.

Choose a basis $X_1, \ldots, X_d$, $d = \dim\ \mathcal{P} = \dim\ M$, in $\mathcal{P}$. The vector fields X_j commute with L, hence the functions

$$u_j = X_j u, \quad j = 1, \ldots, d$$

are solutions to $(\mathrm{D})_B$ because the boundary data for u in $(\mathrm{DN})_B$ implies zero boundary conditions for u_j in $(\mathrm{D})_B$. Clearly, the system $u_1, \ldots, u_d$ is linearly independent since, otherwise, u is constant on integral manifolds of a vector field of the form $\alpha_1 X_1 + \cdots + \alpha_d X_d$ and, together with $u = \text{const}$ on ∂B, this would imply $u \equiv \text{const}$.

Thus, if we denote $V[u] = \text{span}\,\{u_1, \ldots, u_d\}$ and by V_λ the space of all solutions to $(\mathrm{D})_B$, then $V[u] \subset V_\lambda$ and $d \leq \dim V_\lambda = m_D(B, \lambda)$.

It remains to show that actually $V[u] = V_\lambda$. For this purpose, we represent u in geodesic polar coordinates (r, θ) in M (cf. [He]). In these coordinates, the Laplace–Beltrami operator L has the form

$$L = \Delta(L) + L_{S_r}, \tag{3.1}$$

where $\Delta(L) = \frac{\partial^2}{\partial r^2} + \frac{A'(r)}{A(r)} \frac{\partial}{\partial r}$ is the radial part, $A(r)$ is the area of the sphere $S_r = \partial B(o, r)$ and L_{S_r} is the Laplace–Beltrami operator on S_r. When we use normal coordinates, we see that

$$u_j(x) = X_j u(x) = X_j u(r) = u'(r) X_j r = \alpha(r) \beta_j(\theta),$$

where $\alpha(r) = u'(r)$.

The spherical function $u = u(r)$ satisfies the equation $\Delta(L)u = -\lambda u$ and, differentiating both sides of this equation by r, yields

$$\Delta(L)\alpha + \left(\frac{A'}{A}\right)' \alpha = -\lambda \alpha. \tag{3.2}$$

We also have the condition $\alpha(1) = 0$ because of the Neumann boundary condition in $(\mathrm{DN})_B$.

Comparison of (3.2) and the identity $Lu_j = -\lambda u_j$ with L in the form (3.1) gives

$$L_{S_r} \beta_j = \left[\frac{A'}{A}\right]' \beta_j, \quad j = 1, \ldots, d.$$

The functions β_j are uniquely determined by their values on the unit sphere $S = \partial B$ and for $r = 1$ we have

$$L_S \beta_j = \kappa \beta_j, \quad j = 1, \ldots, d, \tag{3.3}$$

with eigenvalue $\kappa = \left[\frac{A'}{A}\right]'_{r=1}$.

On the other hand, all solutions to the equation

$$L\psi + \lambda\psi = 0 \qquad (3.4)$$
$$\psi \big|_S = 0$$

separate variable r, θ: $\psi(r,\theta) = p(r)q(\theta)$, where p is fixed, and q runs through the eigenspace $V^{(\mu)}$ of the Laplace Beltrami operator on the unit sphere S: $L_S q = \mu q$.

All solutions β_j to (3.3) must be in $V^{(\mu)}$, so $\mu = \kappa$, and as $u_j = \alpha\beta_j$ satisfy (3.4), we have $p = \alpha$.

Thus, $V_\lambda = \{\alpha(r)\, q(\theta) \;\big|\; q \in V^{(\kappa)}\}$. Also,

$$V[u] = \{\alpha(r)\, \beta(\theta) \;\big|\; \beta \in \text{span}\, \{\beta_1, \dots, \beta_d\}\}.$$

In order to conclude that $V[u] = V_\lambda$, it suffices to prove that span $\{\beta_1, \dots, \beta_d\}$ is a K-invariant subspace of $V^{(\kappa)}$, because then these two spaces coincide by the irreducibility of representation of the group K in $V^{(\kappa)}$.

The group K acts only in θ-variables, so the K-invariance of span $\{\beta_1, \dots, \beta_d\}$ is equivalent to the K-invariance of $V[u] = \text{span}\, \{u_1, \dots, u_d\}$.

To check this, we have to prove $u_j \circ k \in V[u]$ for all $j = 1, \dots, d$ and any $k \in K$. By the Taylor formula it suffices to show that $D_1^{m_1} \cdots D_k^{m_k} u_j \in V[u]$, where $D_1, \dots, D_k$ is a basis in $\mathfrak{K}$ and m_i are arbitrary nonnegative integers. In turn, it would follow from the inclusion $D_i u_j \in V[u]$ for all $i = 1, \dots, k$ and $j = 1, \dots, d$.

But, $D_i u_j = D_i X_j u = [D_i, X_j]u + X_j D_i u$. The last summand vanishes because $D_j u = 0$ due to the K-invariance of u. As for the first term, we know that $[D_i, X_j] \in \mathcal{P}$ and, therefore,

$$[D_i, X_j] = \sum_{k=1}^{d} c_{ij}^k X_k$$

for some constants c_{ij}^k. Then $[D_i, X_j]u = \sum_{k=1}^d c_{ij}^k X_k u = \sum_{k=1}^d c_{ij}^k u_k \in V[u]$.

Thus, we have proven

PROPOSITION 3.1: *If $m_{DN}(B, \lambda) = 1$, then $m_D(B, \lambda) = d = \dim M$.*

This Proposition is the second assertion in Theorem 1.

3.3 Now we come to the proof of part 1 of Theorem 1.

The main ingredient of the proof is

LEMMA 3.2: *Let Ω be a bounded domain in M with connected smooth boundary $\partial\Omega$. Suppose that there exists a function $u \in C^1(\overline{\Omega})$ and linearly independent vector fields $T_1, \dots T_k \in \mathfrak{G}$, where k equals the dimension of the Lie algebra $\mathfrak{K}$, $k = \dim \mathfrak{K}$, such that*

(1) *u is real-analytic in Ω, $u \mid_{\partial\Omega} = c = \text{const}$, and $u \not\equiv c$ on Ω,*
(2) *$T_j u = 0$ on Ω, $j = 1, \dots, k$.*

Then, Ω is a geodesic ball, $\Omega = B(a, r)$.

Proof: By (1) there exists $x_0 \in \Omega$ for which $u(x_0) \neq c$. Applying a translation, we can assume $x_0 = o$.

Let us introduce the subspace $\mathfrak{N} \subset \mathfrak{G}$:

$$\mathfrak{N} = \{T \in \mathfrak{G} \mid Tu = 0 \;\text{ on }\; \Omega\}.$$

It is easy to see that $\mathfrak{N}$ is a Lie subalgebra.

Condition (2) implies $\dim \mathfrak{N} \geq k$. Let us consider the closed (connected) Lie subgroup N of G generated by the Lie subalgebra $\mathfrak{N}$. For any $x \in M$ we denote by N_x the N-orbit

$$N_x = \{nx \mid n \in N\}.$$

By construction, $u = \text{const}$ on $N_x \cap \Omega$. In particular, $u = \text{const}$ on $N_o \cap \Omega$. We know that $u(o) \neq c$ and $u \mid_{\partial\Omega} = c$. Therefore, N_o, the orbit of the point o, is disjoint from $\partial\Omega$ and, as N_o is a connected manifold, $N_o \subset \Omega$. Boundedness of Ω implies boundedness of N_o. Proposition 4.4 in Chapter II of [He] states that the orbit N_o, which is equal to the coset space $N/(N \cap K)$, is a closed topological subspace of M. Together with compactness of the group $N \cap K$, this implies that N is a compact Lie subgroup of G.

Then N is contained in a maximal compact subgroup K^*, which is conjugated to K ([He], Th. 2.1), $N \subset K^* = gKg^{-1}$.

For the Lie algebras we have the inclusion $\mathfrak{N} \subset \mathfrak{K}^*$. But $\dim \mathfrak{N} \geq k = \dim \mathfrak{K} = \dim \mathfrak{K}^*$ and, for this reason, $\mathfrak{N} = \mathfrak{K}^*$. This means that the Lie groups N and K^* are locally homomorphic, $N = K^*/H$, where H is a finite normal subgroup of K^*. Thus, any two orbits N_x and K^*_x have a common relatively open subset $\{kx\colon k \in U(e)\}$, where U is a neighborhood of the unity $e \in K^*$ such that $H \cap U(e) = \{e\}$. These two orbits are connected real-analytic manifolds. Thus they must coincide, $N_x = K^*_x$.

Clearly, the orbit K_x^* is a geodesic sphere centered at the point $a = go$. Our final arguments are as follows: the function u is constant both on $\partial\Omega$ and on each sphere N_x. Since $u \not\equiv$ const on $\overline{\Omega}$, we conclude that $\partial\Omega$ coincides with one of the spheres N_x and, correspondingly, Ω is a geodesic ball centered at a. ∎

PROPOSITION 3.3: *The D-multiplicity $m_D(\Omega, \lambda)$ of a DN-eigenvalue λ cannot be less than* $\dim M$.

Proof: We saw that if u is a DN-eigenfunction for Ω with eigenvalue λ, then $X_1u, \ldots, X_du$ are D-eigenfunctions for Ω with the same eigenvalue λ. The inequality $m_D(\Omega, \lambda) < \dim M = d$ would imply $\alpha_1 X_1 u + \cdots + \alpha_d X_d u = 0$ in Ω, for some $\alpha_1, \ldots, \alpha_d$, and then u is constant on integral manifolds of the vector field $\alpha_1 X_1 + \cdots + \alpha_d X_d$. Together with $u\,|_{\partial\Omega} =$ const this would imply $u \equiv$ const, which is impossible. ∎

Now we are ready to prove part 1 of Theorem 1.

Suppose that λ is a DN-eigenvalue for the domain Ω and $m_D(\Omega, \lambda) \leq \dim M = d$.

Choose a basis $D_1, \ldots, D_k$, $k = \dim \mathfrak{K}$, in $\mathfrak{K}$ and a basis $X_1, \ldots, X_d$, $d = \dim \mathcal{P} = \dim M$, in the complementary subspace $\mathcal{P} \subset \mathfrak{G}$.

Let u be an eigenfunction corresponding to λ, i.e., u is a non-constant solution to $(\mathrm{DN})_\Omega$.

The Laplace–Beltrami operator L commutes with the vector fields D_i and X_j, so

$$D_i u \ , \ X_1 u, \ldots, X_d u$$

is a collection of solutions to $(\mathrm{D})_\Omega$ (the Dirichlet boundary data for these functions are consequences of the Neumann boundary data for u).

By the hypothesis on λ, any $d+1$ solutions to $(\mathrm{D})_\Omega$ are linearly dependent and we showed already in Proposition 3.3 that the functions $X_1 u, \ldots, X_d u$ are linearly independent. Hence

$$D_i u = \sum_{j=1}^{d} a_{ij} X_j u$$

for some constants a_{ij}. Thus, if we set

$$T_i = D_i - \sum_{j=1}^{d} a_{ij} X_j,$$

then

$$T_i u = 0, \quad i = 1, \dots, k.$$

It remains to note that the T_i are linearly independent, since $D_i \in \mathfrak{K}, X_j \in \mathcal{P}, \mathfrak{K} \cap \mathcal{P} = \{0\}$ and $\{D_1, \dots, D_k\}$, $\{X_1, \dots, X_d\}$ are bases in $\mathfrak{K}$ and $\mathcal{P}$ respectively. So the hypotheses of Lemma 3.2 are satisfied, and Ω is a geodesic ball. ∎

4. Convergence of eigenfunctions by deformations of the unit ball

In this section, we study the behavior of DN-eigenfunctions of sequences of domains, shrinking to the unit ball. We show that the eigenvalues and eigenfunctions of some subsequences of these domains tend to DN-eigenfunctions of the ball.

4.1 Let $(\Omega_t)_{t \in [0,T)}$ be a DN-deformation of the unit ball B and suppose that $\{t_n\}$ is a sequence of values of the parameter t tending to 0, such that the corresponding eigenvalues $\lambda_n = \lambda_{t_n}$ tend to some limit λ_0, $\lim_{n\to\infty} \lambda_n = \lambda_0$. We denote $\Omega_n = \Omega_{t_n}$ and, by u_n, the solution to $(\mathrm{DN})_{\Omega_n}$ corresponding to the eigenvalue λ_n.

Our purpose is to show that λ_0 is a DN-eigenvalue for B.

LEMMA 4.1: *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. The function u is a solution to $(\mathrm{DN})_\Omega$ with eigenvalue λ if and only if for any $v \in C^2(M)$ the identity holds:*

$$\int_\Omega (u - c) Lv dV = -\lambda \int_\Omega uv dV, \tag{4.1}$$

where dV is the element of G-invariant volume on M.

Proof: If u satisfies the system $(\mathrm{DN})_\Omega$, then $\operatorname{grad} u = u - c = 0$ on the boundary and (4.1) follows from the Green formula and the Product Rule for the divergence.

Conversely, suppose (4.1) holds. If v is an arbitrary function in $C^2(M)$ with compact support in Ω, then the Green formula and (4.1) imply

$$\int_\Omega L(u - c) v dV = \int_\Omega (u - c) Lv dV = -\lambda \int_\Omega uv dV$$

and, therefore, $Lu = -\lambda u$.

In order to obtain boundary conditions for u, take arbitrary $v \in C^2(M)$. By the same argument,

$$\begin{aligned}\int_{\partial\Omega} [(u - c) \operatorname{grad} v - v \operatorname{grad} u] \cdot n \, dS &= \int_\Omega \operatorname{div} [(u - c) \operatorname{grad} v - v \operatorname{grad} u] \, dV \\ &= \int_\Omega [L(u - c) \, v - (u - c) Lv] \, dV = 0,\end{aligned}$$

and varying $v \in C^2(M)$ we obtain $u - c = \operatorname{grad} u = 0$ on $\partial\Omega$. ∎

LEMMA 4.2: *If $u_1, u_2 \in L^2(\Omega) \cap C^2(\Omega)$ satisfy (4.1), then $u_1 = u_2$.*

Proof: We have for $w = u_1 - u_2$:

$$\int_\Omega wLvdV = -\lambda \int_\Omega wvdV, \tag{4.2}$$

where $v \in C^2(M)$ is arbitrary.

Let us extend w to M by setting $w(x) = 0$ for $x \notin \Omega$.

In (4.1) we can replace v by the shifted function $v_g = v \circ g^{-1}$, $g \in G$ and, by a change of variables and G-invariance of the operator L, we obtain (4.2) for $w_{g^{-1}}$. Integrating over G with a smooth weight gives us a smooth approximation of w in M. So we can assume w is continuous in M.

Now take v in (4.2) to be an arbitrary bounded K-spherical function $v = \varphi_\mu$ on M, $L\varphi_\mu = \mu\varphi_\mu$ and if $\mu \neq \lambda$ we have $\int_\Omega w\varphi_\mu dV = 0$.

Thus $\hat{w}(\mu) = 0$ for all $\mu \neq \lambda$, where $\hat{w}$ is the K-spherical Fourier transform. Then the K-average

$$w^\#(x) = \int_K w(kx)\, dk$$

is identically zero and, in particular, $w(o) = w^\#(o) = 0$. Since (4.2) is true for all translates w_g, we conclude $w \equiv 0$. ∎

4.2 Let $\Omega_n, u_n, \lambda_n, \lambda_0$ be as in 4.1. Let us normalize u_n by the condition

$$\int_{\Omega_n} |u_n|^2 dV = 1.$$

The function u_n satisfies $Lu_n + \lambda_n u_n = 0$ in Ω_n and $u_n - c_n = \operatorname{grad} u_n = 0$ on $\partial\Omega_n$. We extend u_n to the whole space M by

$$u_n^*(x) = \begin{cases} u_n(x), & x \in \overline{\Omega}_n, \\ 0, & x \notin \overline{\Omega}_n. \end{cases}$$

Then u_n^* belongs to the unit sphere in $L^2(M)$ and, due to its compactness in the weak topology, there exists a subsequence u_{n_k} which converges in the weak topology,

$$u_{n_k}^* \xrightarrow{w} u_0^* \in L^2(M).$$

LEMMA 4.3: *The limit function* $u_0 = u_0^* \mid_{\overline{B}}$ *belongs to* $C^2(B) \cap C^1(\overline{B})$ *and satisfies*

$$Lu_0 + \lambda_0 u_0 = 0 \quad \text{in } B, \quad u_0 - c_0 = \operatorname{grad} u_0 = 0 \quad \text{on } \partial B,$$

for some constant c_0, *which is a limit point of* $\{c_{n_k}\}$.

Proof: By Lemma 4.1 we have

$$\int_M u_n^* LvdV - c_n \int_{\Omega_n} LvdV = -\lambda_n \int_M u_n^* vdV \tag{4.3}$$

for any $v \in L^2(M) \cap C^2(M)$. Here $u_n \mid_{\partial\Omega_n} = c_n$. We claim that the sequence c_n is bounded. If not, then for some subsequence, $|c_{n_k}| \to \infty$. Dividing (4.3) by c_n and letting $n \to \infty$, we would arrive, using the boundedness of λ_n and the L^2-boundedness of u_n^*, at $\int_B LvdV = 0$, which cannot be true for an arbitrary v. Thus, c_n is bounded and, therefore, there exists a convergent subsequence $c_{n_k} \to c_0$.

For the sake of simplicity, renumerate all the sequences under consideration and let us assume that $c_n \to c_0$. Now we can let $n \to \infty$ in (4.3) and we obtain

$$\int_M u_0^* LvdV - c_0 \int_B LvdV = -\lambda_0 \int_M u_0^* vdV.$$

Since all u_n^* vanish out of Ω_n, $u_n^* \xrightarrow{w} u_0^*$ and domains Ω_n tend to B, the u_0^* also vanishes in $M \setminus B$ and we have

$$\int_B u_0 LvdV - c_0 \int_B LvdV = -\lambda_0 \int_B u_0 vdV \tag{4.4}$$

where $u_0 = u_0^* \mid_B$.

Take $v \in C^2(M)$ with compact support in B. By the Green formula, $\int_B LvdV = 0$ and

$$\int_B u_0 LvdV = -\lambda_0 \int_B u_0 v \, dV \tag{4.5}$$

which means that u_0 is a weak solution of the equation $Lu_0 + \lambda_0 u_0 = 0$ in the unit ball B. By the ellipticity of the Laplace–Beltrami operator, u_0 is a classical solution, i.e., $u_0 \in C^2(B)$.

Now, for any $k \in K$ the shifted function $u_0 \circ k$ also satisfies (4.5) and Lemma 4.2 asserts that $u_0 \circ k = u_0$. Therefore, u_0 is K-invariant and is a solution to the equation

$$\Delta(L)u_0 + \lambda_0 u_0 = 0 \quad \text{in } B.$$

But all such solutions are known [GV]:

$$u_0(r) = {}_2F_1\left(\frac{1}{2}(\rho + i\mu)\ ,\ \frac{1}{2}(\rho - i\mu)\ ,\ \alpha + 1, -\sinh^2 kr\right),$$

where ${}_2F_1$ is the hypergeometric function, $\alpha = \frac{d}{2} - 1$, ρ and k are real parameters uniquely associated to the symmetric space M, and $k^2(\rho^2 + \mu^2) = \lambda_0$. The only information we want to derive from this expression is that u_0 can be extended to a function in $C^1(\overline{B})$.

Now (4.4) and Lemma 4.1 complete the proof. ∎

Lemma 4.3 does not as yet mean that λ_0 and u_0 are a corresponding DN-eigenvalue and DN-eigenfunction, since we did not check yet that $u_0 \not\equiv \text{const}$.

The following Lemma provides a tool to prove this.

LEMMA 4.4: *Let G be a bounded domain in M with a smooth boundary, and $\{\psi_k\}_{k=0}^{\infty}$ be an orthonormal basis in $L^2(G)$ of eigenfunctions of L with boundary conditions on ∂G.*

Let $\Omega \Subset G$ be a subdomain with smooth boundary $\partial\Omega$, and for $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ suppose that

$$\Delta u + \lambda u = 0, \quad u - c = \operatorname{grad} u = 0 \quad \text{on } \partial\Omega.$$

Then,

$$\sum_{k=1}^{\infty} \mu_k^2 \left(1 - \frac{\lambda}{\alpha_k}\right)^2 = c^2 \operatorname{vol}(\Omega)$$

where α_k is the eigenvalue of ψ_k and $\mu_k = \langle u, \psi_k \rangle_{L^2(\Omega)}$.

Proof: Let us extend u to the function $u^* \in L^2(G)$ by:

$$u^*(x) = \begin{cases} u(x), & x \in \overline{\Omega}, \\ 0, & x \in G \setminus \overline{\Omega}. \end{cases}$$

Since $u - c = \operatorname{grad} u = 0$ on $\partial\Omega$ we have by the Green formula

$$\int_{\Omega} [(u - c)L\psi_k - \psi_k Lu]\ dV = 0.$$

But $L\psi_k = -\alpha_k\psi_k$, $Lu = -\lambda u$ and $\int_\Omega \psi_k u dV = \mu_k$, hence we arrive at

$$(\alpha_k - \lambda)\ \mu_k = c \cdot \alpha_k \int_\Omega \psi_k dV.$$

Dividing by α_k (which are positive) we obtain

$$\sum_{k=1}^{\infty} \mu_k^2 \left(1 - \frac{\lambda}{\alpha_k}\right)^2 = c^2 \sum_{k=1}^{\infty} \left[\int_\Omega \psi_k dV\right]^2$$

and the Lemma follows since the right-hand side is equal to $c^2 \operatorname{vol}(\Omega)$ due to the Plancherel formula. ∎

Let us apply Lemma 4.4 to the function u_n^*:

$$\sum_{k=1}^{\infty} \left[\mu_k^{(n)}\right]^2 \left(1 - \frac{\lambda_n}{\alpha_k}\right)^2 = c_n^2\ \operatorname{vol}(\Omega_n), \tag{4.6}$$

where $\mu_k^{(n)} = \langle u_n^*, \psi_k\rangle_{L^2(\Omega)}$, $\{\psi_k\}_{k=1}^{\infty}$ is an orthonormal basis in $L^2(G)$ of eigenfunctions of L with Dirichlet boundary data, and α_k is the corresponding eigenvalue.

The sequence $\lambda_n \to \lambda_0$, as $n \to \infty$, so it is bounded.

Since $\alpha_k \to \infty$, there exists k_0 such that $\lambda_n/\alpha_k \leq \frac{1}{2}$ for $k > k_0$.

Then from Lemma 4.4,

$$\begin{aligned} c_n^2 \operatorname{vol}(\Omega_n) &\geq \sum_{k=k_0+1}^{\infty} \left[\mu_k^{(n)}\right]^2 \left(1 - \frac{\lambda_n}{\alpha_k}\right)^2 \geq \frac{1}{4} \sum_{k=k_0+1}^{\infty} \left[\mu_k^{(n)}\right]^2 \\ &= \frac{1}{4}\left\{1 - \sum_{k=1}^{k_0} \left[\mu_k^{(n)}\right]^2\right\}. \end{aligned} \tag{4.7}$$

We have used here the fact that $\sum_{k=1}^{\infty} \left[\mu_k^{(n)}\right]^2 = \int_G |u_n^*|^2\, dV = 1$.

Now assume, in contradiction with what is claimed, that $u_0 = 0$. This means that $u_n^* \xrightarrow{w} 0$ in $L^2(G)$ and, therefore, for each k the coefficients $\mu_k^{(n)} \to 0$ as $n \to \infty$. Thus, letting $n \to \infty$, we obtain from (4.7) that

$$c_0^2 \operatorname{vol}(B) \geq \frac{1}{4}.$$

But $c_0 = u_0 \big|_{\partial B} = 0$ and we have arrived at a contradiction.

Let us summarize what we have proven in this section as

PROPOSITION 4.5: *If $\lambda_n \to \lambda_0 \neq 0$ as $n \to \infty$ and each λ_n is a DN-eigenvalue for the domain Ω_n, then λ_0 is a DN-eigenvalue for the unit ball B.*

5. Proof of Theorem 2

Let $(\Omega_t)_{t\in[0,T)}$ be a DN-deformation of the unit ball B. Suppose, in contradiction with what is claimed in Theorem 2, that there exists a sequence $t_n \to 0$ such that each $\Omega_n = \Omega_{t_n}$ is *not* a ball. Since the sequence $\lambda_n = \lambda_{t_n}$ is bounded by hypothesis, we can choose a convergent subsequence $\lambda_{n_k} \to \lambda_0$.

Without loss of generality we can assume that the original sequence λ_n converges to λ_0.

Now let us turn to Definition 2.1 of DN-deformations. We can blow up the Laplace–Beltrami operator from Ω_t to B by means of mappings

$$F_t\colon B \to \Omega_t, \quad F_t(x) = F(t,x).$$

Then we obtain the family

$$L_t = L \circ \left(F_t^{-1}\right)^*, \quad \left(F_t^{-1}\right)^* v = v \circ F_t^{-1},$$

of operators of the second order in B.

Since F is a C^2-mapping, the coefficients of the operator L_t are continuous functions of the parameter t. Clearly, the boundary spectral problems for the operator L in Ω_t and for the operator L_t in B are equivalent.

Let us consider the (compact) resolvent $R(L_t,\lambda) = (L_t - \lambda)^{-1}$ which is a continuous function in t as long as λ is a regular value for L_t.

Choose a closed contour γ in the complex plane, which surrounds λ_0 and such that no other D-eigenvalue of L in B is contained in the domain $\Delta(\gamma)$, bounded by γ.

Denote by $P(\gamma,t)$ the projector to the D-eigenspaces of L_t, corresponding to the D-eigenvalues in $\Delta(\gamma)$:

$$P(\gamma,t) = \frac{1}{2\pi i}\int_\gamma R(L_t,\zeta)\, d\zeta$$

(cf. [Kato], [DS]).

If $t=0$, then $L_t \big|_{t=0} = L$ and $P(\gamma,0)$ is just the projector to the eigenspace of λ_0. Since eigenspaces are orthogonal, the dimension of the range of $P(\gamma,t)$ is

$$\dim P(\gamma,t) = \sum_{\lambda\in\Delta(\gamma)} m_D(\Omega_t,\lambda). \tag{5.1}$$

The operator function $P(\gamma, t)$ is continuous in a small enough neighborhood $|t| \leq \delta$ of $t = 0$ and, hence, $\dim P(\gamma, t) \equiv \text{const}$ for $|t| \leq \delta$. And since $t_n \to 0$ and $\lambda_n \to \lambda_0$ as $n \to \infty$, we have for n large enough, $n \geq N$, $|t_n| \leq \delta$ and $\lambda_n \in \Delta(\gamma)$. Each λ_n is a DN-eigenvalue for L in Ω_{t_n} and, by Proposition 3.3, $m_D(\Omega_{t_n}, \lambda_n) \geq \dim M$. Therefore,

$$m_D(B, \lambda_0) = \dim P(\gamma, 0) = \sum_{\lambda \in \Delta(\gamma)} m_D(\Omega_t, \lambda) \geq m_D(\Omega_{t_n}, \lambda_n) \geq \dim M.$$

Thus λ_0 is a D-eigenvalue for the unit ball B, and therefore $\lambda_0 \neq 0$ (if $\lambda_0 = 0$, then by the maximum principle for harmonic functions, any D-eigenfunction for λ_0 is zero and we would have $m_D(B, \lambda_0) = 0$). Then Proposition 4.5 yields that λ_0 is a DN-eigenvalue for B.

Now we can apply Theorem 1, part 2, that says that $m_D(B, \lambda_0) = \dim M$. Then

$$m_D(\Omega_{t_n}, \lambda_n) \leq \sum_{\lambda \in \Delta(\gamma)} m_D(\Omega_{t_n}, \lambda) = m_D(B, \lambda_0) = \dim M$$

and it remains to refer to Theorem 1, part 1 in order to conclude that all Ω_{t_n} are balls when n is large enough. This contradiction completes the proof of Theorem 2. ∎

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